Generalized Wilson-Fisher Critical Points from the Conformal Operator Product Expansion

Ferdinando Gliozzi,[1] Andrea L. Guerrieri,[2,6] Anastasios C. Petkou,[3] and Congkao Wen[4,5,6]
[1]*Dipartimento di Fisica, Universita di Torino and Istituto Nazionale di Fisica Nucleare—sezione di Torino Via P. Giuria 1 I-10125 Torino, Italy*
[2]*Department of Physics, Faculty of Science, Chulalongkorn University, Thanon Phayathai, Pathumwan, Bangkok 10330, Thailand*
[3]*Institute of Theoretical Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece*
[4]*Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA*
[5]*Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy, UCLA, Los Angeles, California 90095, USA*
[6]*I.N.F.N. Sezione di Roma Tor Vergata, Via della Ricerca Scientifica 00133 Roma, Italy*
(Received 13 December 2016; revised manuscript received 10 January 2017; published 9 February 2017)

We study possible smooth deformations of the generalized free conformal field theory in arbitrary dimensions by exploiting the singularity structure of the conformal blocks dictated by the null states. We derive in this way, at the first nontrivial order in the ϵ expansion, the anomalous dimensions of an infinite class of scalar local operators, without using the equations of motion. In the cases where other computational methods apply, the results agree.

DOI: 10.1103/PhysRevLett.118.061601

Introduction.—The remarkable success of the numerical conformal bootstrap [1–4] calls for an analytical explanation of the *unreasonable effectiveness* of the conformal field theory (CFT). It is therefore pertinent to ask whether CFT techniques can reproduce, and eventually surpass in accuracy, the perturbative results for critical indices that have been accumulated over the years for a variety of fixed point theories in different dimensions. This question has been recently asked by Ref. [5] in the context of the ϕ^4 theory in $d = 4 - \epsilon$ dimensions. It was shown there that the critical exponents can be reproduced under the following three assumptions. (I) The perturbative Wilson-Fisher (WF) fixed point is described by a CFT. (II) In the $\epsilon \to 0$ limit correlation, functions approach those of the free theory. (III) The equations of motion describe the transformation of a primary operator in the free theory into a descendant at the WF fixed point. Such an approach has been generalized more recently in Refs. [6–8]; see also [9,10].

Motivated by the same questions, we aim to extend the above ideas to the vast class of generalized free CFTs (GFCFTs) in arbitrary dimensions in order to study their nearby WF fixed points. We show that requiring (II) above makes assumption (III) redundant, since the transformation of free primary operators into descendants of the interacting theory is already dictated by the analytic structure of the null states of the GFCFTs. Then, without the use of equations of motion, we calculate the leading-order critical quantities in a variety of models in diverse dimensions d. For the known cases, our results agree with previous calculations.

The four-point function of arbitrary scalar operators in a generic d-dimensional CFT can be parametrized as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{\Delta_{12}^{+}}|x_{34}|^{\Delta_{34}^{+}}}\left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}^{-}}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}^{-}}, \tag{1}$$

where $\Delta_{ij}^{\pm} = \Delta_i \pm \Delta_j$ and Δ_i is the scaling dimension of $\mathcal{O}_i$, while $u = (x_{12}^2 x_{34}^2/x_{13}^2 x_{24}^2)$ and $v = (x_{14}^2 x_{23}^2/x_{13}^2 x_{24}^2)$ are the cross ratios. This corresponds to inserting the conformal operator product expansion (OPE) in the direct channel, in which case the function $g(u,v)$ can be expanded in terms of conformal blocks $G_{\Delta,\ell}^{a,b}(u,v)$, i.e., eigenfunctions of the quadratic Casimir operator of $\mathrm{SO}(d+1,1)$:

$$g(u,v) = \sum_{\Delta,\ell} c_{\Delta,\ell} G_{\Delta,\ell}^{a,b}(u,v). \tag{2}$$

Here $a = -\Delta_{12}^{-}/2$ and $b = \Delta_{34}^{-}/2$, and Δ and ℓ are the scaling dimensions and the spin of the primary operators that are exchanged. When Δ takes some particular values, the generic conformal blocks are singular. It has been shown in Ref. [3] that the singularities are poles and they are associated to null states as

$$G_{\Delta,\ell}^{a,b} = F_{\Delta,\ell}^{a,b} + \sum_k R_k^{a,b} \frac{G_{\Delta'_k,\ell'_k}^{a,b}}{\Delta - \Delta_k}, \tag{3}$$

where F is an entire function of Δ. The poles in (3) reflect the contribution of a null state at $[\Delta_k, \ell_k]$; the form of the residue tells us that if $R_k^{a,b} \neq 0$, such a null state has a descendant at $[\Delta'_k, \ell'_k]$. The explicit formulas for $[\Delta_k, \ell_k]$, $R_k^{a,b}$, and $[\Delta'_k, \ell'_k]$ can be found in Ref. [3]. Another, equivalent, way to obtain them is to consider the following expansion, valid for arbitrary a, b, d, and δ:

$$u^\delta = \sum_{\tau=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(-1)^\ell (2\nu)_\ell}{\tau!\ell!(\nu)_\ell(\nu+\ell+1)_\tau} c^{a,b}_{\delta,\tau,\ell} G^{a,b}_{2\delta+2\tau+\ell,\ell}(u,v), \tag{4}$$

where $(x)_y = [\Gamma(x+y)/\Gamma(x)]$ is the Pochhammer symbol and $\nu = (d/2) - 1$. The coefficients above are given by

$$c^{a,b}_{\delta,\tau,\ell} = \frac{\prod_{i=a,b}(i+\delta)_{\ell+\tau}(i+\delta-\nu)_\tau}{(\Delta-1)_\ell(\Delta-\nu-\tau-1)_\tau(\Delta-2\nu-\tau-\ell-1)_\tau} \tag{5}$$

with $\Delta = 2\delta + 2\tau + \ell$. The three families of simple poles of $c^{a,b}_{\delta,\tau,\ell}$ correspond exactly to the null states of (3). Requiring the cancellation of these poles in the rhs of (4) yields the explicit form of the residues of (3) [11] which precisely produce all the results of Ref. [3]. In the following, we need only the dimensions and the residues of those scalar null states having a scalar descendant. Their scaling dimensions Δ_k and those of the scalar descendants Δ'_k are labeled by an integer k:

$$\Delta_k = \frac{d}{2} - k, \qquad \Delta'_k = \frac{d}{2} + k, \qquad k = 1, 2, \ldots, \tag{6}$$

and the corresponding residue is given by

$$R^{a,b}_k = -\frac{(-1)^k k \prod_{c=\pm a,\pm b}(\frac{d}{4} - \frac{k}{2} - c)_k}{(k!)^2(\frac{d}{2} - k - 1)_{2k}}. \tag{7}$$

Notice that $\Delta_k + \Delta'_k = d$, which means the above operators are *shadows* of each other.

We want to apply such universal properties of the conformal blocks to the study of smooth deformations of GFCFTs. Scalar GFCFTs are constructed by a single elementary scalar field ϕ_f with dimension Δ_{ϕ_f}. Its two-point function is given by $\langle\phi_f(x_1)\phi_f(x_2)\rangle = 1/x_{12}^{2\Delta_{\phi_f}}$, and the three-point function of ϕ_f vanishes. All other correlation functions, either of ϕ_f or of its composites, are given by Wick contractions. We are particularly interested in the cases in which $\Delta_{\phi_f} = (d/2) - k$. They correspond to GFCFTs that have a Lagrangian description as massless free theories with a ∂^{2k} kinetic term that can be coupled to the stress energy tensor [12–15]. The $k = 1$ case corresponds to the free canonical theories. Some GFCFTs with $k > 1$ play an important role in the study of $1/N$ expansions in Gross-Neveu and $O(N)$ vector models in high dimensions [12–14]. Applying the Wick contractions to the product $\phi_f^p(x)\phi_f^{p+1}(y)$ generates the OPE [see Eq. (11)]

$$[\phi_f^p] \times [\phi_f^{p+1}] = \sqrt{p+1}[\phi_f] + \cdots. \tag{8}$$

Throughout the Letter, we normalize the operator as $[\phi_f^p] = \phi_f^p/\sqrt{p!}$ such that the two-point function takes the following simple form:

$$\langle[\phi_f^p](x_1)[\phi_f^p](x_2)\rangle = \frac{1}{(x_{12}^2)^{\Delta_{\phi_f^p}}}, \tag{9}$$

where $\Delta_{\phi_f^p}$, the dimension of the composite operator $[\phi_f^p]$, is simply $p\Delta_{\phi_f}$ for a free theory. Since $\Delta_{\phi_f} = (d/2) - k$, there is a possible contribution of the null state (6); however, according to (7), the residue $R^{a,b}_k$ is zero. We will deform this free theory by replacing $\phi_f^p \to \phi^p$ with

$$\Delta_{\phi^p} = \Delta_{\phi_f^p} + \gamma_p = \Delta_{\phi_f^p} + \gamma_p^{(1)}\epsilon + \gamma_p^{(2)}\epsilon^2 + \cdots. \tag{10}$$

We also deform the relevant OPE coefficients such that they have a smooth $\epsilon \to 0$ limit to their corresponding free values. As a consequence, now $R^{a,b}_k \propto \epsilon^2 \neq 0$, and in the conformal block $G^{a,b}_{\Delta_\phi,0}$ a new subrepresentation associated to a descendant of dimension $\Delta'_k = (d/2) + k$ appears. If in the free-field theory OPE there is a primary operator of dimension Δ'_k, we say that the theory is smoothly deformable. In the interacting theory, such an operator is promoted to a descendant of the null state. The matching of $R^{a,b}_k$ with the OPE coefficient of the free theory yields an equation for the anomalous dimensions of the involved operators. The systematic study of these equations allows us to find the first nontrivial anomalous dimensions of ϕ^p for a huge class of GFCFTs, in arbitrary dimensions. Our results include the known cases of Wilson-Fisher points near canonical free-field theories. Equations (18), (19), and (23)–(27) show our main findings.

The generalized Wilson-Fisher critical points.—Our general strategy to find smooth deformations of GFCFTs consists in constructing a suitable four-point function of a generalized free theory at arbitrary space dimension d and switching on an interaction by simply replacing $d \to d - \epsilon$ and $\Delta_{\phi_f^p} \to \Delta_{\phi^p}$ according to (10). As discussed in the previous section, this procedure defines actually a smooth deformation of the nearby free OPE only if it does not generate new primary operators in the limit $\epsilon \to 0$. In this way, we are essentially studying generalized Wilson-Fisher critical points. Given that we start with arbitrary Δ and d, we have a huge theory space to explore. In the following, we will discuss just a few nontrivial examples that include most of the known results obtained for the Wilson-Fisher critical points by other methods but also yield some new results.

The ϕ^{2n} critical points.—Consider the generic free OPE

$$[\phi_f^p] \times [\phi_f^{p+1}] = \sum_{n=1}^{p+1} \lambda_{p,p,2n-1}[\phi_f^{2n-1}] + \text{spinning blocks}, \tag{11}$$

with the OPE coefficients being

$$\lambda_{p,p,2n-1} = B_{2n-1,n}\sqrt{\frac{p+1}{(2n-1)!}}(p-n+2)_{n-1}, \tag{12}$$

where $B_{n,m}$ is the binomial coefficient. For $\Delta_{\phi_f} = d/2 - k$, $k = 1, 2, 3, \ldots$, inserting (11) into the direct channel of the four-point function $\langle \phi^p \phi^{p+1} \phi^p \phi^{p+1} \rangle$, one obtains an expansion in terms, among others, of scalar conformal blocks of the type $G^{-a_f, a_f}_{\Delta_{\phi_f^{2n-1}}}(u, v)$ with $a_f = \Delta_{\phi_f}/2$. When we smoothly deform the theory, each OPE coefficient should be modified with a term which vanishes in the limit of $\epsilon \to 0$. Most importantly, the operator ϕ^{2n-1} becomes a descendant, which should be removed from the OPE. As we discussed in the previous section, the conformal block in the interacting theory $G^{-a,a}_{\Delta_\phi}$ has a singularity with a residue proportional to the conformal block $G^{-a,a}_{(d/2)+k}$ which is precisely the missing conformal block of the operator ϕ^{2n-1} in the free theory in the limit $\epsilon \to 0$. For the interacting theory, we have $a = a_f + \gamma_{p+1} - \gamma_p$ and $\Delta_\phi = \Delta_{\phi_f} + \gamma_1$, with γ_n being the anomalous dimensions of ϕ^n as defined in (10). Matching the operator dimensions requires $d = 2nk/(n-1)$. Using then the explicit result for the residue in (7), the matching condition gives

$$\lambda^2_{p,p,1} \frac{R^{-a,a}_k}{\Delta_\phi - \Delta_{\phi_f}} = \lambda^2_{p,p,2n-1}. \tag{13}$$

Using the explicit form of $R^{-a,a}_k$ in (7) and performing the ϵ expansion yields

$$(p+1) \frac{(-1)^{k+1} (\frac{d}{2} - k)_k}{4k (\frac{d}{2})_k} \frac{(\gamma^{(1)}_{p+1} - \gamma^{(1)}_p)^2 \epsilon^2 + \cdots}{\gamma^{(1)}_1 \epsilon + \gamma^{(2)}_1 \epsilon^2 + \cdots} = \lambda^2_{p,p,2n-1}. \tag{14}$$

Since the rhs is finite, this immediately implies $\gamma^{(1)}_1 = 0$. The above matching equation then leads to the following recursion relation:

$$\gamma^{(1)}_{p+1} - \gamma^{(1)}_p = \kappa(k,n)(p - n + 2)_{n-1}, \tag{15}$$

with

$$\kappa^2(k,n) = \frac{(-1)^{k+1} 4k B^2_{2n-1,n} (\frac{nk}{n-1})_k}{(2n-1)! (\frac{k}{n-1})_k} \gamma^{(2)}_1, \tag{16}$$

where we used $d = 2nk/(n-1)$. The apparent sign ambiguity of $\kappa(k,n)$ will be solved in a moment. Using the fact that $\gamma_0 = 0$, the recursion relation gives

$$\gamma^{(1)}_p = \frac{\kappa(k,n)}{n} (p - n + 1)_n. \tag{17}$$

The crucial observation is that the operator ϕ^{2n-1} that becomes a descendant of ϕ has a fixed dimension, namely, $\frac{d}{2} + k$. From this fact, we deduce that its anomalous dimension is $\gamma^{(1)}_{2n-1} = n - 1$. Using this, we fix $\kappa(k,n) = n(n-1)/(n)_n$, and plugging this back into (17) yields

$$\gamma^{(1)}_p = \frac{(n-1)}{(n)_n} (p - n + 1)_n, \tag{18}$$

which is interestingly independent of k. From (16), we then obtain the anomalous dimension of ϕ at the order of ϵ^2:

$$\gamma^{(2)}_1 = (-1)^{k+1} 2 \frac{n (\frac{k}{n-1})_k}{k (\frac{nk}{n-1})_k} (n-1)^2 \left(\frac{(n!)^2}{(2n)!} \right)^3. \tag{19}$$

A few comments are in order. In the case of $k = 1$, namely, for the canonical scalar, $\gamma^{(2)}_1$ reduces to $2(n-1)^2 [(n!)^2/(2n)!]^3$, which is a well-known multicritical result [16]: For $n = 2$, it corresponds to the ϕ^4 theory in $d = 4 - \epsilon$, while $n = 3$ to the ϕ^6 theory in $d = 3 - \epsilon$. More generally, when $k > 1$, it is a smooth deformation of a scalar GFCFT with $\Delta_\phi = (d/2) - k = (k/n - 1)$ in $d = (2nk/n - 1)$. For $k > 1$, we have assumed that we turn on only one possible marginal deformation of the form ϕ^{2n}; in principle, one may have marginal interactions with derivatives. Notice also that $k > 1$ allows us to study multicritical, but nonunitary, theories in integer dimensions $d > 6$.

$O(N)$ models.—Here we apply our method to theories with global $O(N)$ symmetry. We consider scalars ϕ_i, $i = 1, 2, \ldots, N$, with $\Delta_{\phi_i} = d/2 - k$ in $d = 3k$ and $d = 4k$ as our examples. We denote $\sum_i \phi_i \phi_i = \phi^2$ and consider the following free OPEs:

$$\begin{aligned}
&[\phi_i(\phi^2)^{p-1}] \times [(\phi^2)^p] \\
&\quad = \sqrt{\frac{2p}{N}} \left([\phi_i] + \sqrt{\frac{(6p + N - 4)^2}{2(N+2)}} [\phi_i \phi^2] \right. \\
&\qquad \left. + \sqrt{\frac{2(10p + 3N - 8)^2 (p-1)^2}{(N+4)(N+2)}} [\phi_i (\phi^2)^2] \right) + \cdots, \\
&[(\phi^2)^p] \times [\phi_i (\phi^2)^p] \\
&\quad = \sqrt{\frac{2p + N}{N}} \left([\phi_i] + \sqrt{\frac{18p^2}{N+2}} [\phi_i \phi^2] \right. \\
&\qquad \left. + \sqrt{\frac{2p^2 (10p + N - 6)^2}{(N+2)(N+4)}} [\phi_i (\phi^2)^2] \right) + \cdots.
\end{aligned} \tag{20}$$

Again, when the interaction turns on, the operator $[\phi_i \phi^2]$ becomes a descendant at $d = 4k - \epsilon$, while $[\phi_i (\phi^2)^2]$ becomes a descendant at $d = 3k - \epsilon$. For $d = 4k - \epsilon$ the matching condition is given by the following two relations:

$$\begin{aligned}
\frac{(-1)^{k+1} (k)_k}{4k (2k)_k} \frac{(\gamma^{(1)}_{p-1,1} - \gamma^{(1)}_{p,0})^2}{\gamma^{(2)}_{0,1}} &= \frac{(6p + N - 4)^2}{2(N+2)}, \\
\frac{(-1)^{k+1} (k)_k}{4k (2k)_k} \frac{(\gamma^{(1)}_{p,1} - \gamma^{(1)}_{p,0})^2}{\gamma^{(2)}_{0,1}} &= \frac{18p^2}{N+2}.
\end{aligned} \tag{21}$$

Here we denote the anomalous dimension of $(\phi^2)^p$ as $\gamma_{p,0} = \gamma^{(1)}_{p,0}\epsilon + \gamma^{(2)}_{p,0}\epsilon^2 + \cdots$, and similarly for $\phi_i(\phi^2)^p$, we have $\gamma_{p,1} = \gamma^{(1)}_{p,1}\epsilon + \gamma^{(2)}_{p,1}\epsilon^2 + \cdots$. Removing $\gamma^{(1)}_{p,0}$ from (21), we obtain a recursion relation of $\gamma^{(1)}_{p,1}$, whose solution is given by [17]

$$\gamma^{(1)}_{p,1} = \beta(k,N)p(6p+N+2), \tag{22}$$

with $\beta^2(k,N) = (-1)^{k+1}[2k(2k)_k/(k)_k(N+2)]\gamma^{(2)}_{0,1}$. As before, $p = 1$ is a special case where $\phi^i\phi^2$ becomes a descendant with a fixed conformal dimension $d/2 + k$; this leads to $\gamma^{(1)}_{1,1} = 1$. Thus, we have

$$\gamma^{(2)}_{0,1} = (-1)^{k+1}\frac{(k)_k}{2k(2k)_k}\frac{N+2}{(N+8)^2}. \tag{23}$$

Plugging this result back into (22), we also obtain

$$\gamma^{(1)}_{p,1} = \frac{p(6p+N+2)}{N+8}, \quad \gamma^{(1)}_{p,0} = \frac{p(6p+N-4)}{N+8}. \tag{24}$$

A similar analysis for the theory in $d = 3k - \epsilon$ gives

$$\gamma^{(2)}_{0,1} = (-1)^{k+1}\frac{(k/2)_k}{8k(3k/2)_k}\frac{(N+2)(N+4)}{(3N+22)^2}, \tag{25}$$

as well as

$$\gamma^{(1)}_{p,1} = \frac{10p+3N+2}{3(22+3N)}p(2p-1),$$
$$\gamma^{(1)}_{p,0} = \frac{2(10p+3N-8)}{3(22+3N)}p(p-1). \tag{26}$$

The results of $\gamma^{(2)}_{0,1}$ for $k = 1$ agree with known results using other methods [18,19]. By considering different correlators, one can obtain anomalous dimensions for other operators [11]. For instance, for the anomalous dimension $\gamma^{(1)}_{p,1,1}$ of the symmetric traceless tensor $\phi_i\phi_j(\phi^2)^p - (1/N)\delta_{ij}(\phi^2)^{p+1}$, we have

$$\gamma^{(1)}_{p,1,1} = \frac{2+p(8+N+6p)}{N+8}, \qquad d = 4k-\epsilon,$$
$$\gamma^{(1)}_{p,1,1} = \frac{2p[2+3p(N+4)+10p^2]}{3(3N+22)}, \quad d = 3k-\epsilon. \tag{27}$$

Conclusion.—In this Letter, we applied general properties of CFTs, and, in particular, the singularity structure of generic conformal blocks, to study the possible smooth deformations of GFCFTs in arbitrary dimensions. Our nontrivial results correspond to generalized Wilson-Fisher critical points. The examples presented in this Letter include general classes of multicritical points and $O(N)$ invariant theories. Our corresponding results for theories with multiple deformations will appear in the longer version of this work [11]. Combining the OPE structure with universal properties of certain scalar null states, we derived, at the first nontrivial order in the ϵ expansion, the anomalous dimensions of an infinite class of scalar local operators. In the particular cases where other computational methods were applied, the results agree. Our method allows us to put huge classes of critical theories under a unified calculation scheme. We also remark that, unlike the usual conformal bootstrap program, neither crossing symmetry nor unitarity play crucial roles in our scheme. Therefore, we believe that our method, properly extended, can be useful to study nonunitary critical systems that are relevant in physics. As a final remark, we observe that we have considered even marginal deformations of the form ϕ^{2n}. When the deformation is odd, namely, ϕ^{2n-1}, our method appears to be less powerful. Furthermore, it is of great interest to consider more general deformations (such as turning on multiple marginal interactions) and more general operators (such as operators with spins). More details on these questions as well as on the calculations presented in this Letter will appear in the longer version of our work [11].

Our work has been strongly inspired by some unpublished notes of Yu Nakayama on smooth deformations of the free-field theory in $d = 6 - \epsilon$ dimensions. We thank him for sharing with us his notes. We also thank M. Bianchi and D. Simmons-Duffin for helpful discussion. F. G. thanks the OIST of Okinawa where part of this work was done. The work of C. W. is supported by DOE Grant No. DE-SC0010255. A. C. P. thanks CPHT École Polytechnique, for its warm hospitality during the final stages of this work. The work of A. L. G. is funded under a CUniverse research promotion project by Chulalongkorn University (grant reference CUAASC).

[1] S. El-Showk, M. F. Paulos, and D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012).
[2] S. El-Showk, M. F. Paulos, and D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, J. Stat. Phys. **157**, 869 (2014).
[3] F. Kos, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 11 (2014) 109.
[4] F. Kos, D. Poland, and D. Simmons-Duffin, and A. Vichi, J. High Energy Phys. 08 (2016) 036.
[5] S. Rychkov and Z. M. Tan, J. Phys. A **48**, 29FT01 (2015).
[6] P. Basu and C. Krishnan, J. High Energy Phys. 11 (2015) 040.
[7] K. Nii, J. High Energy Phys. 07 (2016) 107.
[8] C. Hasegawa and Y. Nakayama, arXiv:1611.06373.
[9] R. Gopakumar, A. Kaviraj, and K. Sen, and A. Sinha, arXiv:1609.00572.
[10] R. Gopakumar, A. Kaviraj, and K. Sen, and A. Sinha, arXiv:1611.08407.

[11] F. Gliozzi, A. Guerrieri, and A. C. Petkou, and C. Wen (to be published).
[12] K. Diab, L. Fei, and S. Giombi, I. R. Klebanov, and G. Tarnopolsky, J. Phys. A **49**, 405402 (2016).
[13] A. Guerrieri, A. C. Petkou, and C. Wen, J. High Energy Phys. 09 (2016) 019.
[14] H. Osborn and A. Stergiou, J. High Energy Phys. 06 (2016) 079.
[15] C. Brust and K. Hinterbichler, arXiv:1607.07439.
[16] Jean Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, International Series of Monographs on Physics (Clarendon, Oxford, 2002).
[17] There is a sign ambiguity here, which is fixed by requiring it reproduce the results of $N = 1$, namely, the single scalar case we considered in the previous section.
[18] J. Hofmann, Nucl. Phys. **B350**, 789 (1991).
[19] J. S. Hager, J. Phys. A **35**, 2703 (2002).